

Critical Property of Geometric Phase in the Dicke Model

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We obtain the ground-state energy level and associated geometric phase in the Dicke model analytically by means of the Holstein-Primakoff transformation and the boson expansion approach in the thermodynamic limit. The non-adiabatic geometric phase induced by the photon field is derived with the time-dependent unitary transformation. It is shown that the quantum phase transition characterized by the non-analyticity of the geometric phase is remarkably of the first-order. We also investigate the scaling behavior of the geometric phase at the critical point, which can be measured in a practical experiment to detect the quantum phase transition.

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Berry phase[1] with the relaxation of its original restriction conditions[2, 3, 4, 5] has been extensively generalized along various directions [6, 7, 8, 9]. Recently, the geometric phase (GP) has been regarded as an essential way to implement operation of a universal quantum logic gate in quantum computing[10, 11, 12, 13, 14, 15] and as an important tool to detect quantum phase transition (QPT) in the XY model[16, 17], which describes a structural change in the properties of the ground state energy spectrum associated with the variation of a controlling parameter[18] and originates from singularity of the energy spectrum[19]. A single 1/2-spin driven by a classical rotating magnetic field is a well studied model to generate the Berry phase. Recently a quantized magnetic field has been considered and shown to induce the GP which reduces to the standard Berry phase in the semiclassical limit[20, 21]. The generation of this framework of Berry phase to many-body two-level-atom system interacting with a single bosonic mode known as Dicke model[22, 23] is certainly of interest particularly in connection with the QPT. The Dicke model displays an interesting super-radiant phenomenon describing the collective and coherent behaviors of atoms and exhibits the QPT from the normal to the super-radiant phases[24, 25, 26] induced by the variation of the coupling strength between the atom and field. It has been known that the atomic ensemble in the normal phase is collectively unexcited while is macroscopically excited with coherent excitations in the so-called super-radiant phase. The QPT has been related to the emergence of chaos in a corresponding classical Hamiltonian[27, 28] and the logarithmic diverges of the von Neumann entropy at the critical transition point which describes the quantum entanglement between the atoms and field[29, 30, 31].

In this paper we derive the non-adiabatic GP of many-body atom-system driven by the quantized field for the first time. The ground-state and corresponding GP are obtained with the Holstein-Primakoff transformation[32] and the boson expansion approach[33] in the thermodynamic limit. We demonstrate that the GP can serve as a critical criteria to characterize the QPT. The scaling behavior of the GP at the critical point is also studied.

The Dicke model of N two-level atoms in a single-mode light field is given in the rotating-wave approximation by

$$H = \omega a^\dagger a + \sum_{i=1}^N \left[\frac{\omega_0}{2} \sigma_z^i + \frac{\lambda}{\sqrt{N}} (\sigma_+^i a + \sigma_-^i a^\dagger) \right] \quad (1)$$

where a , a^\dagger are the photon annihilation and creation operators; σ_+^j and σ_-^j are the pseudo-spin operators for the j -th atom defined as $\sigma_\pm^j = \sigma_x^j \pm i\sigma_y^j$ with σ_x and σ_y being the Pauli matrices. λ denotes the coupling strength between the atom and field and ω_0 is the energy difference between two levels of the atom in the unit $\hbar = 1$. The prefactor $1/\sqrt{N}$ is inserted to have a finite free energy per atom in the thermodynamical limit $N \rightarrow \infty$. The Hamiltonian eq.(1) is actually considered in a rotating frame along with the light field and becomes

$$H'(t) = R(t) H R^\dagger(t) - i R(t) \frac{dR^\dagger(t)}{dt}, \quad (2)$$

in the laboratory frame with the time-dependent unitary transformation

$$R(t) = \exp[-i\varphi(t)a^\dagger a], \quad (3)$$

where $\varphi(t) = \omega t$ denotes the rotating angle. The non-adiabatic GP induced by the light field can be found by solving the time-dependent Schrödinger equation

$$i \frac{d|\Psi'_n(t)\rangle}{dt} = H'(t) |\Psi'_n(t)\rangle, \quad (4)$$

and the result is

$$\begin{aligned} \gamma_n &= i \int_0^{2\pi} \langle \Psi'_n(\varphi(t)) | \frac{d}{d\varphi} |\Psi'_n(\varphi(t))\rangle d\varphi \\ &= 2\pi \langle \Psi_n | a^\dagger a | \Psi_n \rangle. \end{aligned} \quad (5)$$

where $|\Psi'_n(\varphi(t))\rangle = R(t) |\Psi_n\rangle$ with $|\Psi_n\rangle$ being the eigenstate of the Hamiltonian H such that $H|\Psi_n\rangle = E_n|\Psi_n\rangle$, which is equal to the Berry phase formula in Refs.[20, 21]. We are interested in the critical property of the GP associated with the ground state $|\Psi_0\rangle$ of

the many-body system. Using collective giant-spin operators $J_z = \sum_{i=1}^N \sigma_z^i$, $J_\pm = \sum_{i=1}^N \sigma_\pm^i$, which satisfy the usual SU(2) commutation relations $[J_z, J_\pm] = \pm J_\pm$ and $[J_+, J_-] = 2J_z$ with total spin quantum number $j = N/2$, the Hamiltonian eq.(1) becomes

$$H = \omega a^\dagger a + \frac{\omega_0}{2} J_z + \frac{\lambda}{\sqrt{N}} (J_+ a + J_- a^\dagger). \quad (6)$$

The system undergoes a QPT at a critical value of the atom-field coupling strength λ_c (to be determined below) from the normal phase when $\lambda < \lambda_c$ to the super-radiant phase when $\lambda > \lambda_c$ in the thermodynamical limit. By means of the boson expansion approach and the Holstein-Primakoff transformation of the collective angular momentum operators defined as $J_+ = b^\dagger \sqrt{N - b^\dagger b}$, $J_- = \sqrt{N - b^\dagger b} b$ and $J_z = (b^\dagger b - N/2)$, where the new boson operators satisfy the commutation relation $[b, b^\dagger] = 1$ [32]. With this transformation the Hamiltonian becomes

$$H = \omega a^\dagger a + \frac{\omega_0}{2} (b^\dagger b - N/2) + \frac{\lambda}{\sqrt{N}} (b^\dagger \sqrt{N - b^\dagger b} a + \sqrt{N - b^\dagger b} b b a^\dagger) \quad (7)$$

Following the procedure of Ref.[28], we introduce shifting boson operators c^\dagger and d^\dagger with properly scaled auxiliary parameters α and β such that

$$c^\dagger = a^\dagger + \sqrt{N}\alpha, \quad d^\dagger = b^\dagger - \sqrt{N}\beta \quad (8)$$

to evaluate the critical transition point λ_c and the ground state energies of both two phases. It should be noticed that the displacements, $c^\dagger = a^\dagger - \sqrt{N}\alpha$ and $d^\dagger = b^\dagger + \sqrt{N}\beta$, also lead to the same result except for the change of sign. Expanding Hamiltonian eq.(7) with the displacement boson operators c^\dagger and d^\dagger as power series of $1/\sqrt{N}$ we can obtain

$$H = NH_0 + N^{1/2}H_1 + N^0H_2 + \dots \quad (9)$$

with

$$\begin{aligned} H_0 &= \omega \alpha^2 + \frac{\omega_0}{2} (\beta^2 - \frac{1}{2}) - 2\lambda \alpha \beta \sqrt{k}, \\ H_1 &= (-\omega \alpha + \lambda \beta \sqrt{k})(c^\dagger + c) + [\frac{\omega_0 \beta}{2} - \lambda \alpha \sqrt{k}(1 - 2\beta^2)](d^\dagger + d), \\ H_2 &= \omega c^\dagger c + \frac{\omega_0}{2} d^\dagger d + \lambda [\sqrt{k} d^\dagger - \frac{\beta^2}{2\sqrt{k}}(d^\dagger + d)]c \\ &\quad + \lambda [\sqrt{k} d - \frac{\beta^2}{2\sqrt{k}}(d^\dagger + d)]c^\dagger + \frac{\lambda \alpha \beta}{2\sqrt{k}} [4d^\dagger d + (d^\dagger)^2 + d^2 + \frac{\beta^2}{2k}(d^\dagger + d)^2], \end{aligned}$$

where $k = 1 - \beta^2$. The first term of H gives rise to the Hartree-Bogoliubov ground state energy[34]

$$E_0 = \langle \Psi_0 | NH_0 | \Psi_0 \rangle = \begin{cases} -N\omega_0/4, & \lambda < \lambda_c \\ -N[\frac{\lambda^2}{4\omega}(1 - \delta^2) + \frac{\omega_0 \delta}{4}], & \lambda > \lambda_c \end{cases}, \quad (10)$$

and the critical value

$$\lambda_c = \sqrt{\omega_0 \omega / 2} \quad (11)$$

where $|\Psi_0\rangle = |0\rangle_{at} |0\rangle_{ph}$ is the vacuum of the quasi-boson operators such that $c|0\rangle_{ph} = 0$ and $d|0\rangle_{at} = 0$, (correspondingly $a|0\rangle_{ph} = \sqrt{N}\alpha|0\rangle_{ph}$, $b|0\rangle_{at} = -\sqrt{N}\beta|0\rangle_{at}$) with $\delta = \omega\omega_0/2\lambda^2$. The auxiliary parameters

$$\alpha = \begin{cases} 0, & \lambda < \lambda_c \\ \lambda \sqrt{1 - \delta^2} / 2\omega, & \lambda > \lambda_c \end{cases} \quad (12)$$

$$\beta = \begin{cases} 0, & \lambda < \lambda_c \\ \sqrt{(1 - \delta)/2}, & \lambda > \lambda_c \end{cases} \quad (13)$$

are determined from minimizing the ground state energy eq.(10). In the normal phase ($\lambda < \lambda_c$) the system is essentially in the lower energy state and is only microscopically excited, whereas above the transition point both the field and the atomic ensemble acquire macroscopic excitations.

The GP of the many-body system associated with the ground state $|\Psi_0\rangle$ is obtained in terms of the quasi-boson operator eq.(8) as

$$\gamma_0 = 2\pi \langle \Psi_0 | a^\dagger a | \Psi_0 \rangle = \begin{cases} 0, & \lambda < \lambda_c \\ \frac{\pi N}{2\omega^2} (\lambda^2 - \frac{\lambda^4}{\chi^2}), & \lambda > \lambda_c \end{cases}. \quad (14)$$

The scaled GP γ_0/N and its first-order derivative with respect to the coupling parameter λ is shown in Fig.1 with the resonant condition $\omega = \omega_0 = 1$. It can be seen that the GP vanishes when $\lambda < \lambda_c$ and increases abruptly with λ when $\lambda > \lambda_c$ indicating a first-order phase transition at the critical point λ_c . It is apparent that GP can be used to detect the quantum criticality in systems described by the Dicke model. The quantum criticality is also shown to relate the quantum entanglement governed by the von Neumann entropy which has a cups-like behavior at the critical transition point[29, 30]. While the GP behaves as a step function at the critical point.

The scaling behavior of GP at the critical point can be found in the thermodynamical limit as

$$\frac{\gamma_0}{N}(\lambda \rightarrow \lambda_c) = \frac{2\pi\lambda_c}{\omega^2} |\lambda - \lambda_c|. \quad (15)$$

On the other hand, the first-order derivative of the GP diverges linearly with the atom number N at the transition point λ_c as

$$\lim_{N \rightarrow \infty} \frac{d\gamma_0}{d\lambda} \big|_{\lambda=\lambda_c} = \frac{2\pi\lambda_c}{\omega^2} N, \quad (16)$$

which is different from the logarithmic divergence of the first-order derivative of the Berry phase in XY model[16, 17].

In recent years it has been shown that some nano-systems, for example the semiconductor quantum-dots, coupled with a high-quality single-cavity-mode can be a promising candidate for implementing the Hamiltonian of Dicke model and the realistic parameter values are given by $\lambda \sim 0.1 - 1.5$ [35, 36, 37] in the unit ω_0 . While ω_0 is of order of the underlying bulk band-gap with the value-range from 1.5 eV in GaAs-like semiconductor down to 0.1 eV in narrow-gap semiconductor. The value of λ is

also variable in a wide range from weak to strong coupling regimes based on current nanotechnology. The GP can be observed directly by measuring the mean photon number out of the cavity.

In conclusion, we have investigated the critical property of the non-adiabatic GP derived for the first time in terms of time-dependent unitary transformation in single-mode super-radiant Dicke model, which displays the first-order QPT. The GP is shown to be proportional to the mean photon number, and therefore is calculated analytically for the ground-state of the Dicke model in the thermodynamical limit. We also provide the scaling behavior of GP as a probe to test the QPT.

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- [1] M.V.Berry, Proc. R. Soc. London, Ser.A **392**, 45 (1984).
 - [2] Y. Aharonov and J.Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
 - [3] J.Anandan, Phys. Lett. A **133**, 171 (1988).
 - [4] J.Samuel and R. Bhandari, Phys. Rev. Lett. **60**, 2339 (1988).
 - [5] A. K. Pati, Phys. Rev. A **52**, 2576 (1995).
 - [6] E. Sjöqvist et al., Phys. Rev. Lett. **85**, 2845 (2000).
 - [7] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **90**, 160402 (2003).
 - [8] X. X. Yi, L. C. Wang and T. Y. Zheng, Phys. Rev. Lett. **92**, 150406 (2004).
 - [9] B. Wu, J. Liu, and Q. Niu, Phys. Rev. Lett. **94**, 140402 (2005).
 - [10] P. Zanardi and M.Rasetti, Phys. Lett. A **264**, 94 (1999).
 - [11] G. Falci et al., Nature (London) **407**, 355 (2000).
 - [12] S. L. Zhu, and Z. D. Wang, Phys. Rev. Lett. **89**, 097902 (2002).
 - [13] S. L. Zhu, and Z. D. Wang, Phys. Rev. A **67**, 022319 (2003).
 - [14] S. L. Zhu, and Z. D. Wang, Phys. Rev. Lett. **91**, 187902 (2003).
 - [15] S. L. Zhu, Z. D. Wang, and P. Zanardi, Phys. Rev. Lett. **94**, 100502 (2005).
 - [16] A. C. M. Carollo and J. K. Pachos, Phys. Rev. Lett. **95**, 157203 (2005).
 - [17] S. L. Zhu, Phys. Rev. Lett. **96**, 077206 (2006).
 - [18] S. Sachdev, *Quantum Phase transitions* (Cambridge University Press, Cambridge, 1999).
 - [19] C. N. Yang and T. D. Lee, Phys. Rev. **87**, 410 (1952).
 - [20] I. Fuentes-Guridi, A. Carroll, S. Bose, V. Vedral, Phys. Rev. Lett. **89**, 220404 (2002).
 - [21] A. Carollo, I. Fuentes-Guridi, M. F. Santos, and V. Vedral, Phys. Rev. Lett. **92**, 020402 (2004).
 - [22] R. H. Dicke, Phys. Rev. **93**, 99 (1954).
 - [23] M. Gross, and S. Haroche, Phys. Rep. **93**, 301 (1982).
 - [24] K. Hepp and E. H. Lieb, Ann. Phys. (N. Y.) **76**, 360 (1973).
 - [25] Y. K. Wang and F. T. Hioes, Phys. Rev. A **7**, 831(1973).
 - [26] F. T. Hioes, Phys. Rev. A **8**, 1440(1973).
 - [27] C. Emary and T. Brandes, Phys. Rev. Lett. **90**, 044101 (2003).
 - [28] C. Emary and T. Brandes, Phys. Rev. E **67**, 066203 (2003).
 - [29] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. Lett. **92**, 073602 (2004).
 - [30] N. Lambert, C. Emary, and T. Brandes, Phys. Rev. A **71**, 053804 (2005).
 - [31] V. Bužek, M. Orszag, and M. Roško, Phys. Rev. Lett. **94**, 163601 (2005)..
 - [32] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1949).
 - [33] A. Klein and E. R. Marshalek, Rev. Mod. Phys. **63**, 375 (1991).
 - [34] A. Dzhiyev et. al., Phys. Rev. C **69**, 014318 (2004).
 - [35] H. Haug, and S.W. Koch, *Quantum Theory of the Optical and Electronic Properties of Semiconductors* (World Scientific, Singapore, 2004).
 - [36] E. Hagley et al., Phys. Rev. Lett. **79**, 1 (1997).
 - [37] A. Rauschenbeutel et al., Science **288**, 2024 (2000).

Figure Caption

Fig.1 The scaled ground-state geometric phase γ_0/N as a function of the coupling parameter λ and its first-order derivative in the resonant condition $\omega = \omega_0 = 1$.